

Schrödinger cat state of trapped ions in harmonic and anharmonic oscillator traps

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We examine the time evolution of a two level ion interacting with a light field in harmonic oscillator trap and in a trap with anharmonicities. The anharmonicities of the trap are quantified in terms of the deformation parameter τ characterizing the q -analog of the harmonic oscillator trap. Initially the ion is prepared in a Schrödinger cat state. The entanglement of the center of mass motional states and the internal degrees of freedom of the ion results in characteristic collapse and revival pattern. We calculate numerically the population inversion $I(t)$, quasi-probabilities $Q(t)$, and partial mutual quantum entropy $S(P)$, for the system as a function of time. Interestingly, small deformations of the trap enhance the contrast between population inversion collapse and revival peaks as compared to the zero deformation case. For $\beta = 3$ and 4, (β determines the average number of trap quanta linked to center of mass motion) the best collapse and revival sequence is obtained for $\tau = 0.0047$ and $\tau = 0.004$ respectively. For large values of τ decoherence sets in accompanied by loss of amplitude of population inversion and for $\tau \sim 0.1$ the collapse and revival phenomenon disappear. Each collapse or revival of population inversion is characterized by a peak in $S(P)$ versus t plot. During the transition from collapse to revival and vice-versa we have minimum mutual entropy value that is $S(P) = 0$. Successive revival peaks show a lowering of the local maximum point indicating a dissipative irreversible change in the ionic state. Improved definition of collapse and revival pattern as the anharmonicity of the trapping potential increases is also reflected in the Quasi- probability versus t plots.

I. INTRODUCTION

Two-level ions trapped in a harmonic or anharmonic trap have turned into an important tool for understanding the time evolution of non classical states. Experimentally harmonic oscillator traps have been realized and various types of non-classical states of ions in trapped systems constructed [1,2] with sufficient control over amplitudes, relative phases etc. of the component states. It is now possible to detect [3] experimentally constructed non-classical states and measure the statistics of the quantum motion of the center of mass of the ion. The non-classical states constructed and detected include the Fock states, thermal states, Shrodinger cat states and squeezed states. The possibility of manipulating the entangled states of particle systems provides the basis for applications like quantum computations and quantum communications. On the other hand, these experimental advances have paved a way to an understanding of some of the basic aspects of quantum theory. A controlled manipulation of these states requires precise control of the Hamiltonian of the system. The form of the trap potential is usually controlled by the geometry of the system, the laser field intensities and relative phases. As such anharmonic traps with different potential shapes should also be viable experimentally. The measurement of dynamical behavior of non-classical states in harmonic and anharmonic traps and a study of anharmonicity related new features makes an interesting study. Harmonically trapped and laser driven ion system has been studied in many contexts such as nonlinear coherent states [4,5], vibrionic Jaynes-Cummings interactions [6], nonlinear Jaynes-Cummings interactions [7,8], and generation of even and odd coherent states [9]. The trapped ion-laser system is in some sense an experimental realization of Jaynes Cummings model [10], just like the micromaser cavity experiments. A study of the dynamics of two-level atom coupled to the q -analog of a single mode of the bosonic cavity field [11] indicates that q deformation of Heisenberg algebra may correspond to some effective nonlinear interaction of the cavity mode. In our earlier work [12] on ions in harmonic and anharmonic traps, we examined the collapses and revivals of the population inversion and quasi-probabilities as a function of time for the initial state of the system having ion in it's ground state and the center of mass motion described by a coherent state. This study brought out some very interesting new features in the dynamics of the system. In the present work, we consider a two-level ion initially in a Schrödinger cat state and examine the collapse and revival of population inversion and quasi-probabilities with a focus on the coherence of the system and anharmonicities of the trap. We also examine the time evolution of partial mutual entropy of the system to understand the improvement in collapse and revival

pattern of the system for characteristic values of parameter quantifying the trap anharmonicities. The motivation for studying these states is provided by the possible use in quantum computation and quantum communication where entanglement of ionic internal states with center of mass motion may be used to encode information [13–16]. In Ref. [12] as well as in this work, the anharmonicity of the trap potential is modeled through a q -analog harmonic oscillator trap. Since Macfarlane [17] and Biedenharn [18] discussed q -analog of harmonic oscillator, a lot of work using these in various fields of physics has been done. In our efforts to understand the physical nature of q -deformations in the context of pairing effect in nuclei [19–21], the q -deformation is found to simulate the nonlinearities of the problem caused by unaccounted part of the residual interaction. In the present context, with the parameter q being used to quantify the anharmonicity of the trap potential, the underlying energy spectrum of vibrational states of ion as well as the laser ion interaction contain dependence on the number operator for vibrational quanta and the parameter q . For small values of q , the q -analog harmonic oscillator is essentially an anharmonic oscillator with x^6 anharmonicities [22]. Nonlinear couplings of vibrational modes of ion have been studied theoretically [23–25] with the nonlinear effects modeled in the laser-ion interaction Hamiltonian through Hermitian operators with strong dependence on the number operator for vibrational quanta. The underlying trap in these studies is a harmonic oscillator one. We may point out that for ion in a harmonic oscillator trap and non-linear ion-laser coupling expressed through q -deformed operators with the q values used in our work, the system behavior continues to be very close to the case of ion in a harmonic trap and linear ion-laser couplings.

We start with a description of the trapped ion-laser system Hamiltonian in section I for the cases of ion in a Harmonic Oscillator trap and ion in a q -analog harmonic oscillator trap. In section II the initial state and the equations governing its time evolution are given. Population inversion and quasi-probability are defined in section III. In section IV, we discuss the concept of partial mutual entropy for the system and the numerical results are analyzed in section V.

II. THE HAMILTONIAN

A. Harmonic Oscillator trap

The system that consists of the two level ion moving in a harmonic oscillator trap potential and interacting with a classical single-mode light field of frequency ω_l is described by the Hamiltonian,

$$H = \frac{1}{2}\hbar\omega(a^\dagger a + aa^\dagger) + \frac{1}{2}\hbar\Delta\sigma_z + \frac{1}{2}\hbar\Omega(F\sigma^+ + F^*\sigma^-) \quad (1)$$

where a and a^\dagger are the usual creation and destruction operators for the oscillator, $\Delta = \omega_a - \omega_l$, is the detuning parameter and Ω is the Rabi frequency of the system. The pseudo-spin operators of the two-level ion, σ^\pm and σ_z operate on state vectors $|g\rangle$ and $|e\rangle$ where indices g and e stand for the ground and excited states of the two level ion. The operator F stands for [6] $\exp(ikx) = \exp[i\epsilon(a^\dagger + a)]$, where k is the wave vector of the light field and x position quadrature of the center of mass. The parameter $\epsilon = \sqrt{\frac{E_r}{E_t}}$ is a function of the ratio of the recoil energy of the ion $E_r = \frac{\hbar^2 k^2}{2m}$ and the characteristic trap quantum energy $E_t = \hbar\omega = \frac{v}{a_0}$. The second term in the Hamiltonian refers to the energy associated with internal degrees of freedom of the ion, whereas the third term is the interaction of the ion with the light field. By using a special form of Baker-Hausdorff Theorem the operator F may be written as a product of operators i.e.e we use the equality

$$F = \exp[i\epsilon(a^\dagger + a)] = e^{(\frac{\epsilon^2}{2}[a^\dagger, a])} e^{i\epsilon a^\dagger} e^{i\epsilon a}. \quad (2)$$

For harmonic oscillator trap one has

$$[a, a^\dagger] = 1; \quad Na^\dagger - a^\dagger N = a^\dagger; \quad Na - aN = -a. \quad (3)$$

The physical processes implied by the various terms of the operator

$$F = \left[\exp\left(\frac{-\epsilon^2}{2}\right) \sum_n \frac{(i\epsilon)^n a^{\dagger n}}{n!} \sum_k \frac{(i\epsilon)^k a^k}{k!} \right], \quad (4)$$

may be divided into three categories. The terms for $n > k$ correspond to an increase in energy linked with the motional state of center of mass of the ion by $(n - k)$ quanta. The terms with $n < k$ represent destruction of $(k - n)$ quanta of energy thus reducing the amount of energy linked with the center of mass motion. For $(n = k)$, we have diagonal contributions. The contribution from a particular term containing operators $a^{\dagger n} a^k$ is determined by the coefficient $\exp\left(\frac{-\epsilon^2}{2}\right) \left[\frac{(i\epsilon)^{n+k}}{n!k!} \right]$.

B. q -analog harmonic oscillator trap

Next we consider a single two level ion having ionic transition frequency ω_a in a quantized q -analog quantum harmonic oscillator trap (q -deformed harmonic oscillator trap) interacting with a single mode travelling light field. The creation and annihilation operators for the trap quanta satisfy the following commutation relations,

$$AA^\dagger - qA^\dagger A = q^{-N} \quad ; \quad NA^\dagger - A^\dagger N = A^\dagger \quad ; \quad NA - AN = -A. \quad (5)$$

The operators A and A^\dagger act in a Hilbert space with basis vectors $|n\rangle$, $n = 0, 1, 2, \dots$, given by,

$$|n\rangle = \frac{(A^\dagger)^n}{([n]_q!)^{\frac{1}{2}}} |0\rangle \quad (6)$$

such that the number operator $N|n\rangle = n|n\rangle$. The vacuum state is $A|0\rangle = 0$. We define here $[x]_q$ as

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \quad (7)$$

and the q -analog factorial $[n]_q!$ is recursively defined by

$[0]_q! = [1]_q! = 1$ and $[n]_q! = [n]_q[n-1]_q!$. It is easily verified that

$$A^\dagger |n\rangle = [n+1]_q^{\frac{1}{2}} |n+1\rangle \quad ; \quad A |n\rangle = [n]_q^{\frac{1}{2}} |n-1\rangle \quad (8)$$

and N is not equal to $A^\dagger A$. Analogous to the harmonic oscillator one may define the q -momentum and q -position coordinate

$$P_q = i\sqrt{\frac{m\hbar\omega}{2}}(A^\dagger - A) \quad ; \quad X_q = \sqrt{\frac{\hbar}{2m\omega}}(A^\dagger + A). \quad (9)$$

The q -analog harmonic oscillator Hamiltonian is given by

$$H_{qho} = \frac{1}{2}\hbar\omega(AA^\dagger + A^\dagger A) \quad (10)$$

with eigenvalues

$$E_n = \frac{1}{2}\hbar\omega([n+1]_q + [n]_q). \quad (11)$$

We note that the trap states are not evenly spaced, the energy spacing being a function of deformation. Besides that as we move up in the number of vibrational quanta in the states the spacing between successive states increases. We choose $q = e^\tau$, where τ is a real valued parameter. For $\tau = 0.0$ the harmonic oscillator trap is recovered. As such the parameter τ is a measure of the extent to which the harmonic oscillator potential trap is deformed. Using the Taylor expansion of the q -number $[n]_q$ in terms of powers of τ^2 the energy eigenvalues may be rewritten for small τ as

$$E_n = \hbar\omega \left[\left(n + \frac{1}{2}\right) \left(1 - \frac{\tau^2}{24}\right) + \left(n + \frac{1}{2}\right)^3 \frac{\tau^2}{6} + \dots \right]. \quad (12)$$

Bonatsos et. Al [26] have shown that the potential giving a spectrum similar to that of Eq. (12) up to the order τ^2 looks like

$$V(x) = \left(\frac{1}{2} - \frac{\tau^2}{8}\right)x^2 + \frac{\tau^2}{120}x^6, \quad (13)$$

which is an anharmonic oscillator with x^6 anharmonicities [22].

Using a nonlinear map given by Curtright and Zachos [27] one can express H_{qho} in terms of the operators a and a^\dagger of the harmonic oscillator. To make this point clear we express the operators A and A^\dagger as

$$A = a f(N) \quad ; \quad A^\dagger = f(N) a^\dagger \quad (14)$$

where $f(N) = \left(\frac{[N]_q}{N}\right)^{\frac{1}{2}}$ and $N = a^\dagger a$. We can also verify that

$$f(N)a^\dagger = a^\dagger f(N+1) ; f(N)a = af(N-1) \quad (15)$$

The Hamiltonian of Eq. 10 can be rewritten as

$$H_{qho} = \hbar\omega \left(f(N+1)^2 + f(N)^2 \right) \left(a^\dagger a + \frac{1}{2} \right) \quad (16)$$

which can be interpreted as a harmonic oscillator Hamiltonian with a frequency $\omega_q(N)$ that depends on the quantum number n of the state in question.

The quadratures x and p are related to P_q and X_q through

$$\begin{aligned} P_q &= i\sqrt{\frac{m\hbar\omega}{2}} \left((f(N) - f(N+1)) \frac{x}{\sqrt{2}} - (f(N) + f(N+1)) \frac{ip}{\sqrt{2}} \right) \\ X_q &= \sqrt{\frac{\hbar}{2m\omega}} \left((f(N) + f(N+1)) \frac{x}{\sqrt{2}} - (f(N) - f(N+1)) \frac{ip}{\sqrt{2}} \right) \end{aligned} \quad (17)$$

C. Hamiltonian for an ion interacting with light in a q -analog harmonic oscillator trap

The Hamiltonian for an ion interacting with light in a q -analog harmonic oscillator trap may now be written as

$$H_q = \frac{1}{2}\hbar\omega(AA^\dagger + A^\dagger A) + \frac{1}{2}\hbar\Delta\sigma_z + \frac{1}{2}\hbar\Omega(F_q\sigma^+ + F_q^\dagger\sigma^-)$$

where by analogy with Eq. 4 we choose

$$F_q = e^{\left(-\frac{|\epsilon|^2}{2}\right)} e^{i\epsilon A^\dagger} e^{i\epsilon A}, \quad (18)$$

which in the limit $q \rightarrow 1$ reduces to F , and can be expanded as [28],

$$F_q = e^{\left(-\frac{|\epsilon|^2}{2}\right)} \sum_{n=0}^{\infty} \frac{(i\epsilon)^n A^{\dagger n}}{n!} \sum_{k=0}^{\infty} \frac{(i\epsilon)^k A^k}{k!}. \quad (19)$$

Various terms in the expansion of this operator represent processes which might result in transitions of the center of mass from a given motional state, in the q -analog trap, to another, while loosing or gaining energy. The coefficient of the operators $A^{\dagger n} A^k$ is again $\exp\left(\frac{-\epsilon^2}{2}\right) \left[\frac{(i\epsilon)^{n+k}}{n!k!}\right]$, the same as that in the corresponding term with operators $a^{\dagger n} a^k$ in Eq. 4. Intuitively this is the correct way of representing the interaction of the ion and the laser in a q -analog harmonic oscillator trap. This form of the operator shows that the energy exchange of the center of mass motion of the ion occurs as the ion moves up or down in the trap. Since $E_{n+1} - E_n$ is n dependent, it implies an n dependent entanglement of the center of mass motion and the internal degrees of freedom of the two level atom. Using Eq. 14, we can rewrite Eq. 19 as

$$F_q = e^{\left(-\frac{|\epsilon|^2}{2}\right)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(i\epsilon)^{n+k} (\epsilon f(N)a^\dagger)^n (\epsilon a f(N))^k}{n!k!}. \quad (20)$$

Comparing with F we notice that the effective lamb Dicke parameter in a q -analog trap, for the loss and gain of motional state energy in an interaction process, is $\epsilon f(N)$ where N is the number operator. With the q -analog of Glauber coherent state defined as

$$|\alpha\rangle_q = \frac{1}{\sqrt{\exp_q|\alpha|^2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_q!}} |n\rangle,$$

the calculated value of ${}_q \langle \alpha | f(N) | \alpha \rangle_q$ is 1.0004, for the choice $\alpha = 4$, $\epsilon = 0.05$ and $\tau = 0.003$.

It is important to recall at this point that the operator $(A^\dagger + A)$ is not a self adjoint operator. The vectors $|m\rangle$ are analytic vectors of the operators $(A^\dagger)^n A^k$ but not of the operators $A^k (A^\dagger)^n$. One can easily verify that the matrix elements $\langle m | e^{\left(\frac{|\epsilon|^2}{2}\right) \sum_{l=0}^{\infty} \frac{(i\epsilon)^k A^k}{k!} \sum_{n=0}^{\infty} \frac{(i\epsilon)^n A^{\dagger n}}{n!} | m \rangle$ are not well defined. On the other hand the matrix elements of the operator in Eq. 18 are well defined as discussed in Ref. [28]. From the form of the operator in Eq. 19 it is clear that the product $F_q \sigma^+$ contains all the processes in which the atom is excited from the ground state to the excited state while at the same time some of the energy quanta in the center of mass motion are lost or gained. It is also important to note that X_q is not the true position coordinate but is related to x and p in some complex way. As such the operator $\exp[i\epsilon(A^\dagger + A)]$ can not be used to represent the laser-ion interaction.

The expression for the matrix element of the operator F_q between the states $\langle m |$ and $|n\rangle$ for $m \leq n$ is given by,

$$\langle m | F | n \rangle = \frac{e^{-\frac{|\epsilon|^2}{2}} (i\epsilon)^{n-m} [m]_q^{\frac{1}{2}}!}{[n]_q^{\frac{1}{2}}!} \sum_{k=0}^m \frac{(\epsilon)^{2k} (-1)^k [n]_q!}{k! (n-m+k)! [m-k]_q!} . \quad (21)$$

III. SCHRÖDINGER CAT STATE

Initially the ion is prepared in a Schrödinger Cat State with equal probabilities of finding the ion in its ground state and in the excited state coupled to a coherent state describing the state of motion of the center of mass of the ion in the anharmonic trap. The experimental technique for producing trapped ion in a state with these characteristics has been described in many experimental papers and such states have been produced [2,29].

The initial state of the trapped ion in a q -analog harmonic oscillator trap can be expressed as

$$\Psi(t=0) = \frac{|g, \beta\rangle_q + e^{i\phi} |e, -\beta\rangle_q}{\sqrt{2}} \quad (22)$$

where $|g, \beta\rangle_q$ is given by

$$|g, \beta\rangle_q = \frac{1}{\sqrt{\exp_q |\beta|^2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{[n]_q!}} |g, n\rangle, \quad (23)$$

and β determines the average number of vibrational quanta linked to the center of mass motion. The relative phase between two possible internal states is determined by ϕ , in the present study we choose $e^{i\phi} = 1$.

The state of the system at a time t ,

$$\Psi(t) = \sum_m g_m(t) |g, m\rangle + \sum_m e_m(t) |e, m\rangle \quad (24)$$

is a solution of the time dependent Schrödinger equation

$$H_q \Psi(t) = i\hbar \frac{d}{dt} \Psi(t). \quad (25)$$

In the state $|g, m\rangle$ ($|e, m\rangle$) the two level ion in its ground (excited) state is coupled to number state $|m\rangle$ of the anharmonic oscillator. The probability amplitudes $g_m(t)$ and $e_m(t)$ satisfy the following set of coupled equations

$$\begin{aligned} i \frac{dg_m(t)}{dt} &= \frac{1}{2} g_m(t) (\omega([m+1]_q + [m]_q) - \Delta) + \frac{1}{2} \Omega \sum_n e_n(t) \langle g, m | \sigma^- F_q^\dagger | e, n \rangle \\ i \frac{de_m(t)}{dt} &= \frac{1}{2} e_m(t) (\omega([m+1]_q + [m]_q) + \Delta) + \frac{1}{2} \Omega \sum_n g_n(t) \langle e, m | F_q \sigma^+ | g, n \rangle \end{aligned} \quad (26)$$

In the limit $q \rightarrow 1$ we recover the system that gives the dynamics of two-level ion in a harmonic oscillator trap. By rescaling ϵ , ω , and t , the parameter Ω can be eliminated from Eq. (26).

IV. POPULATION INVERSION AND QUASI-PROBABILITY

The entanglement of motional degrees of freedom of the center of mass of the ion with its internal degrees of freedom manifests itself in the well known collapse and revival of population inversion. The population inversion is defined as

$$I(t) = P_g(t) - P_e(t), \quad (27)$$

that is the difference between the probability of finding the system in the ground state, $P_g(t)$, and the probability of finding the system in the excited state, $P_e(t)$. The time evolution of population inversion is examined for different values of trap anharmonicities for the cases when coherent state is characterized by parameter $\beta = 3, 4$. The time t is expressed in units of $\frac{\Omega}{2\pi}$.

We have also calculated numerically and plotted as a function of α_r and α_i the Quasi-Probability function $Q(\alpha)$ defined as

$$Q(\alpha) = \frac{1}{\pi_q} \langle \alpha | \rho_{c.m.}(t) | \alpha \rangle_q, \quad (28)$$

where $\rho_{c.m.}(t) = \text{tr}_{ion} \rho(t)$ is the density matrix reduced for degrees of freedom of center of mass motion.

V. QUANTUM MUTUAL ENTROPY

Dynamical behavior of the system can be better understood in terms of quantum mutual entropy. Furuichi and co-workers [30] have applied the concept of quantum mutual entropy to dynamical change of state of the atom in Jaynes-Cummings model (JCM) [10]. The analogy between JCM and ion in a trap system allows us to extend the concept to understand the entanglement of the internal degrees of freedom of ion and its motional degrees of freedom. We next outline the calculation of quantum mutual entropy for the system at hand prepared initially in state given by Eq. (22). It is found that the quantum mutual entropy can be decomposed into a part determined by the relative populations of the system and another that depends on the coherences. The partial quantum mutual entropy used by us in our numerical study is essentially a measure of mutual entropy due to populations developed in the system with the passage of time.

With the initial state of the trapped ion expressed as in (22), The density operator for the internal states of the ion at $t = 0$ is

$$\rho_{ion} = \lambda_1 |g\rangle \langle g| + \lambda_2 |e\rangle \langle e|. \quad (29)$$

The state of motion of the ionic center of mass at $t = 0$ is represented by a coherent state $|\beta\rangle_q$ for ion in ground state and by coherent state $|\beta\rangle_q$ for ion in excited state. where

$$|\beta\rangle_q = \frac{1}{\sqrt{\exp_q(|\beta|^2)}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{[n]_q!}} |n\rangle_q. \quad (30)$$

The state of the system at a time t is given by Eq. (24).

The time evolution of the internal state of the ion can also be represented by a continuous mapping

$$\Lambda_t^{ion} \rho_{ion} = \text{tr}_{c.m.} (U_t \rho_0 U_t) \quad (31)$$

where ρ_0 is the initial state of the system, $U_t = \exp\left(\frac{-iHt}{\hbar}\right)$ a unitary operator and the mapping Λ_t^{ion} maps the initial internal state ρ_{ion} to the current internal state of the ion at time t . We represent the time evolution of initial constituent ionic states $|g\rangle \langle g|$ and $|e\rangle \langle e|$ through the maps

$$\Lambda_t^{ion} |g\rangle \langle g| = \text{tr}_{c.m.} (U_t |g, \beta\rangle \langle g, \beta| U_t) = P_g^{(1)}(t) |g\rangle \langle g| + P_e^{(1)}(t) |e\rangle \langle e| \quad (32)$$

$$+ C_{ge}^{(1)}(t) |g\rangle \langle e| + C_{ge}^{(1)*}(t) |e\rangle \langle g| \quad (33)$$

and

$$\Lambda_t^{ion} |e\rangle \langle e| = \text{tr}_{c.m.} (U_t |e, -\beta\rangle \langle e, -\beta| U_t). \quad (34)$$

The correlation between the initial ionic state and its current state containing information about the time evolution of each constituent ionic states is given in terms of the quantum mutual entropy defined as

$$S_m(\rho^{ion}, \Lambda_t^{ion}) = \sum_{i=1}^2 \lambda_i S(\Lambda_t^{ion} |i\rangle \langle i|, \Lambda_t^{ion} \rho_{ion}) , \quad (35)$$

where the relative quantum entropy

$$S(\Lambda_t^{ion} |i\rangle \langle i|, \Lambda_t^{ion} \rho_{ion}) = \text{tr}(\Lambda_t^{ion} |i\rangle \langle i|) [\log(\Lambda_t^{ion} |i\rangle \langle i|) - \log(\Lambda_t^{ion} \rho_{ion})] \quad (36)$$

and $i = 1$ (2) for ion in ground (excited) state. It measures the relative overlap of the constituent ionic state $|i\rangle$ with the ionic state of the system at a time t for the cases a) where the initial ionic state of the system is state $|i\rangle$ and b) the initial ionic state of the system is given by Eq. (29). The quantum relative entropy is a more general concept than the Von Neumann Entropy being a measure that extends to mixed states. As defined in Eq. (36), it is a measure of how far each entangled state is from its component disentangled state as a result of time evolution. The quantum mutual entropy as defined by Eq. 35 is obtained by summing up the time dependent quantum relative entropies with respect to ionic states that constitute the initial cat state and measures the propagation of information content of the initial state.

Working in the space spanned by basis vectors $\{|u_n\rangle\} \equiv |g, m\rangle, |e, m\rangle, m = 0, 1, \dots, \infty$, we find from Eq. (24) the probabilities of finding the ion in the ground state and the excited state (the populations) to be respectively

$$P_g(t) = \langle g | \text{tr}_{ion} \rho(t) | g \rangle = \sum_n |g_n(t)|^2 \quad (37)$$

and

$$P_e(t) = \langle e | \text{tr}_{ion} \rho(t) | e \rangle = \sum_n |e_n(t)|^2 . \quad (38)$$

Besides that the system at a time t is characterized by the following non-diagonal matrix element of the operator $\text{tr}_{ion} \rho(t)$ (coherence),

$$C_{ge} = \langle g | \text{tr}_{ion} \rho(t) | e \rangle = \sum_n g_n^*(t) e_n(t) . \quad (39)$$

We also define the populations and coherences for the system prepared in initial state $|g, \beta\rangle_q$ ($i = 1$) and the initial state with the ion in the excited state $|e, -\beta\rangle_q$ ($i = 2$). The state of the system at a time t is now given by

$$\Psi^{(i)}(t) = \sum_m g_m^{(i)}(t) |g, m\rangle + \sum_m e_m^{(i)}(t) |e, m\rangle , \quad (40)$$

with the populations defined as

$$P_g^{(i)}(t) = \langle g | \text{tr}_{ion} \rho^{(i)}(t) | g \rangle = \sum_n |g_n^{(i)}(t)|^2 \quad (41)$$

and

$$P_e^{(i)}(t) = \langle e | \text{tr}_{ion} \rho^{(i)}(t) | e \rangle = \sum_n |e_n^{(i)}(t)|^2 . \quad (42)$$

The coherences for these initial states are given by

$$C_{ge}^{(i)}(t) = \langle g | \text{tr}_{ion} \rho^{(i)}(t) | e \rangle = \sum_n (g_n^{(i)}(t))^* e_n^{(i)}(t) . \quad (43)$$

We can verify that in terms of the populations and the coherences defined above the relative quantum entropy, $S(\Lambda_t^{ion} |i\rangle \langle i|, \Lambda_t^{ion} \rho_{ion})$ (Eq. 36) for the system starting in initial state given by Eq. (29) is

$$S(\Lambda_t^{ion} |i\rangle \langle i|, \Lambda_t^{ion} \rho_{ion}) = \left[P_g^{(i)}(t) \log \left(\frac{P_g^{(i)}(t)}{P_g(t)} \right) + P_e^{(i)} \log \left(\frac{P_e^{(i)}(t)}{P_e(t)} \right) \right. \\ \left. + 2 * \text{Re} \left[C_{ge}^{(i)}(t) \log \left(\frac{C_{ge}^{(i)}(t)}{C_{ge}(t)} \right)^* \right] \right] . \quad (44)$$

Substituting Eq. (44) in the definition of quantum mutual entropy for the system, we have

$$S_m = \sum_{i=1,2} \lambda(i) \left[P_g^{(i)}(t) \log \left(\frac{P_g^{(i)}(t)}{P_g(t)} \right) + P_e^{(i)} \log \left(\frac{P_e^{(i)}(t)}{P_e(t)} \right) \right. \\ \left. + 2 * \text{Re} \left[C_{ge}^{(i)}(t) \log \left(\frac{C_{ge}^{(i)}(t)}{C_{ge}(t)} \right)^* \right] \right] . \quad (45)$$

We can split S_m in to two distinct parts, one depending on populations and the other on off diagonal coherences that is

$$S_m = S(P) + S(C) \quad (46)$$

where

$$S(P) = \sum_{i=1,2} \lambda(i) \left[P_g^{(i)}(t) \log \left(\frac{P_g^{(i)}(t)}{P_g(t)} \right) + P_e^{(i)}(t) \log \left(\frac{P_e^{(i)}(t)}{P_e(t)} \right) \right] . \quad (47)$$

We have calculated numerically $S(P)$ the partial mutual quantum entropy for the cases $\beta = 3, 4$ and used the $S(P)$ peaks to pinpoint the t values for which the system is the most correlated and the t values where it shows the least correlation. Essentially $S(P)$ is used as a measure of entanglement of the system.

VI. RESULTS

The population inversion as a function of rescaled time parameter $t(\frac{\Omega t}{2\pi})$ for two level ion interacting with light field in a harmonic oscillator trap and a q -deformed oscillator trap for the cases $\beta = 3, 4$ is displayed in Figures. 1 and 3 for three different values of deformation parameter τ ($q = \exp(\tau)$ with τ real). The set of parameters used in the numerical calculation is $\overline{\omega} = \frac{\omega}{\Omega} = 50$, $\overline{\epsilon} = \frac{\epsilon}{\Omega} = 0.05$, and $\overline{\Delta} = \frac{\Delta}{\Omega} = -50$. The maximum value of n in Eq. (24) is restricted to $n = 32$. For both β values the collapse and revival pattern was found to improve in definition with increase in the anharmonicity of the trap potential, measured by parameter τ . In Figures. 1a and 2a, we display the collapse and revival for ion in a harmonic oscillator trap that is $\tau = 0.0$. Both for $\beta = 3$ and 4 first revival and second revival are easily identified. For $\beta = 3$ the second collapse, which is barely identifiable is succeeded by a region where the inversion pattern becomes diffuse, whereas for $\beta = 4$ the second collapse is followed by another revival which is quite wide. For $\beta = 3$ and 4, the best collapse and revival sequence is obtained for $\tau = 0.0047$ and $\tau = 0.004$ respectively as shown in Figures. 1b and 2b. We can identify four revivals in each case. The collapses and revivals for $\beta = 4$ are remarkably sharp. Further increase in τ results in a loss of contrast between successive revival peaks. Figures. 1c and 2c indicate a rapid loss of phase relationship essential for generating observable collapses and revivals of population inversion between components of the total wave function as the anharmonicity of the trap potential is increased. We have also found that for a large value of τ , different for each β value, not only the collapses and revivals of population inversion vanish but also the time dependence of population inversion disappears in all cases.

In figs. (3-4) we plot the partial quantum mutual entropy $S(P)$ as a function of time for initial states with $\beta = 3$, and 4. A careful examination shows that each collapse as well as revival of population inversion is characterized by a peak in $S(P)$ versus t plot. During the transition from collapse to revival and vice-versa we have minimum mutual entropy value that is $S(P) = 0$. In the q -deformed oscillator trap for the cases $\beta = 3, \tau = 0.0047$ and $\beta = 4, \tau = 0.004$ we notice relatively higher $S(P)$ peaks and well defined transition periods(collapse \leftrightarrow revival) in comparison with the harmonic oscillator trap($\tau = 0.0$). The regions without continuous minimum $S(P)$ do not show well defined collapses or revivals in the population inversion plots. In Table I, the t values for successive peaks in $S(P)$ for the cases $\beta = 3, \tau = 0.0047$ and $\beta = 4, \tau = 0.004$ are displayed. It is evident from figures 3 and 4 that $S(P)$ decreases with the Rabi oscillations. Table I demonstrates that the successive revival peaks show a lowering of the local maximum point indicating a dissipative irreversible change in the ionic state.

Table I. The values of t for successive peaks in $S(P)$ for the cases $\beta = 3, \tau = 0.0047$, $\beta = 4, \tau = 0.0$ and $\beta = 4, \tau = 0.004$.

$\beta = 3, \tau = 0.0047$			$\beta = 4, \tau = 0.0$			$\beta = 4, \tau = 0.004$		
t	$S(P)$	Peak type	t	$S(P)$	Peak type	t	$S(P)$	Peak type
0.0	0.693	Initial state	0.0	0.693	Initial state	0.0	0.693	Initial state
58.6	0.628	revival	85.8	0.333	revival	67.8	0.731	revival
114.0	0.314	collapse	171.4	0.143	collapse	133.2	0.497	collapse
175.6	0.285	revival	266.8	0.106	revival	201.2	0.512	revival
234.0	0.220	collapse	388.2	0.085		266.6	0.359	collapse
295.6	0.181	revival	447.6	0.076		336.8	0.397	revival
360.0	0.234	collapse				404.8	0.334	collapse
415.4	0.175	revival				472.6	0.324	revival

Next we plot in Fig. 5 the Quasi-probabilities for $\beta = 4$ at $t = 0.0, 85.8, 171.4, 266.8, 388.2$ and 447.6 to visualize the time evolution of ionic center of mass in a harmonic oscillator trap ($\tau = 0.0$). As expected, the coherent states $|g, \beta\rangle$ and $|e, -\beta\rangle$, each splits into two and move in a sense opposite to each other in phase space. At $t = 85.8$ there are two somewhat distorted coherent states characterized by almost equal average position coordinate ($\alpha_r \simeq 0$), and momenta with opposite signs. This point corresponds to a revival peak in the population inversion plot and the first peak $S(P)$ versus t plot (Fig. 4a). A second revival similarly occurs at $t = 266.8$. The collapse at $t = 171.4$ corresponds to a situation similar to the initial state. At 388.2 and 447.6 the quasi-probability is spread out in the phase space. For $\tau = 0.004$ quasi-probability contour plots at $t = 0.0, 67.8, 133.2, 201.2, 266.6$, and 336.8 are presented in Fig. 6. Characteristic spreading of the Quasi-probabilities in the phase space is the result of non-linearities of the trap, however the initial state configuration is still recovered signalling collapse and revival phenomenon.

VII. CONCLUSION

Entangled quantum systems present correlations having no classical counterparts and may be exploited to store and transfer information in ways otherwise difficult or impossible. In this work we have examined the effects of trap non-linearity on Schrödinger cat state of a single trapped ion. This particular quantum system may serve in future as a basic element in quantum computation and quantum communication [13–16] or a sensitive detector of decoherence due to interaction with environment. The entanglement of ionic internal states with center of mass motion may be used to encode information in such applications. We have found an enhancement in the contrast between the collapse and revival peaks as the anharmonicity of the trapping potential increases. With further increase in non-linearity not only the collapses and revivals disappear but also the time dependence of population inversion vanishes. This extremely interesting feature is manifested in the Population inversion plots, the Quasi-probability plots as well as the partial mutual entropy plots. The effect is markedly visible in $\beta = 4$ case for the set of parameters used to characterize the ion-laser in trap system. The best collapse and revival sequences are obtained for $\tau = 0.0047$ and $\tau = 0.004$, for $\beta = 3$ and 4 respectively. The reason for this effect is to be found in the deformation dependence of trap energies. The characteristic Rabi Frequency, in addition to the ion-laser interaction now depends on energy separation between trap levels. It can be easily verified that in the lowest approximation, with $\omega \gg \Omega$ and driving field tuned to $\Delta = -\omega$, the Rabi frequency is given by $\mu(n) = \sqrt{\left(\frac{\omega}{2}(\cosh(2(n+1)\tau) - 1)\right)^2 + \Omega^2 \epsilon^2 [n+1]_q}$ which in the limit $\tau \rightarrow 0$ (for $q = e^\tau$) reduces to $\mu(n) = \sqrt{\Omega^2 \epsilon^2 [n+1]}$ as expected. For small values of parameter τ , the factor $\left(\frac{\omega}{2}(\cosh(2(n+1)\tau) - 1)\right)^2$, coming from non-equidistant trap energy spectrum, varies approximately as $(n+1)^4$ and dominates the scene resulting in a high contrast collapse and revival peaks in population inversion. For large values of τ decoherence sets in accompanied by loss of amplitude of population inversion and for $\tau \sim 0.1$ the collapse and revival phenomenon disappear. We use partial quantum mutual entropy as an indicator of time evolution of entanglement. It is obtained by summing up the time dependent quantum relative entropies with respect to ionic states that constitute the initial cat state and retaining the part that depends on populations. Quantum relative entropy is a measure of how far each entangled state moves from its component disentangled state as a result of time evolution. Looking together at the population inversion and partial mutual entropy $S(P)$ plots we verify that the onset of collapse and revival is characterized by $S(P) = 0$. A well defined collapse and revival pattern is characterized by wide $S(P) = 0$ regions. Every collapse or revival corresponds to a peaked $S(P)$. Successive revival peaks show a lowering of the local maximum point which is an indicator of a dissipative irreversible change in the ionic state same being true for $S(P)$ peaks that indicate collapse of the population inversion.

Acknowledgments

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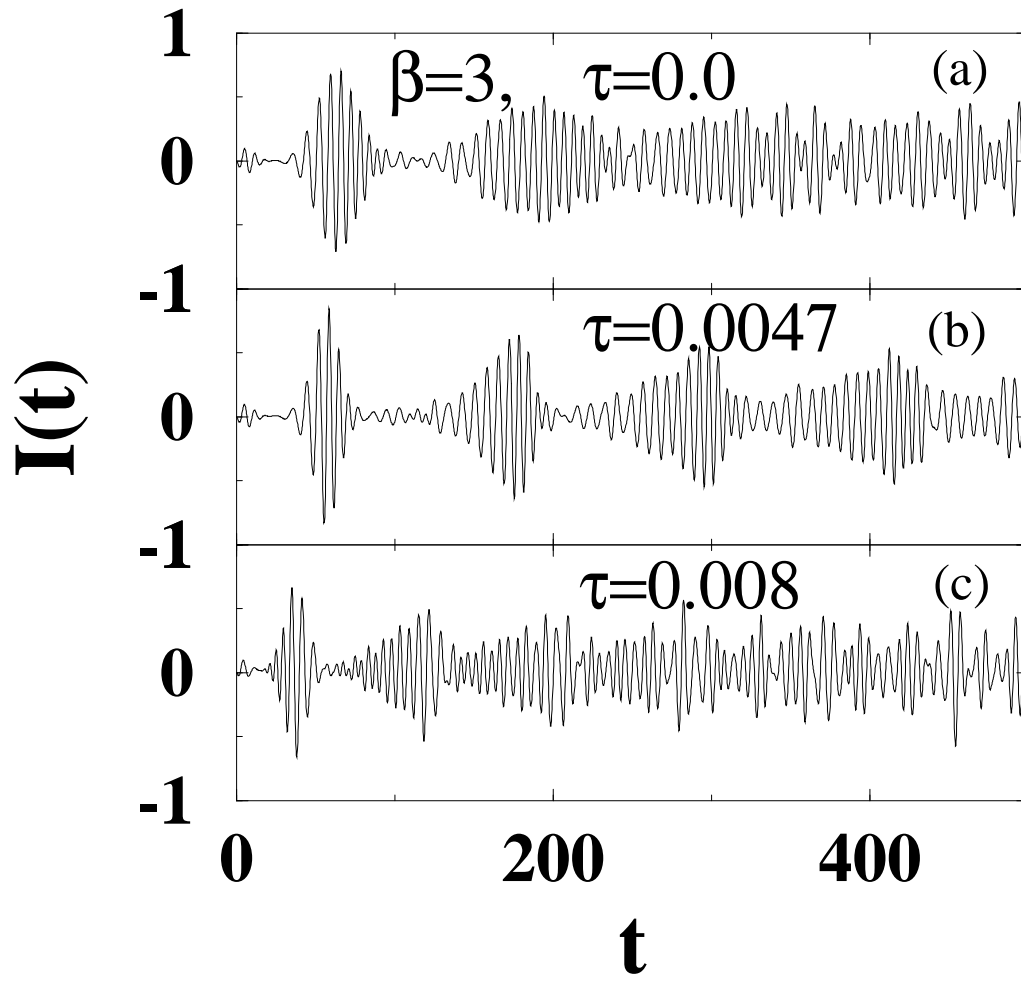


Fig. 1. Population Inversion $I(t)$ versus t for $\beta = 3$ and $\tau = 0.0, 0.0047, 0.008$.

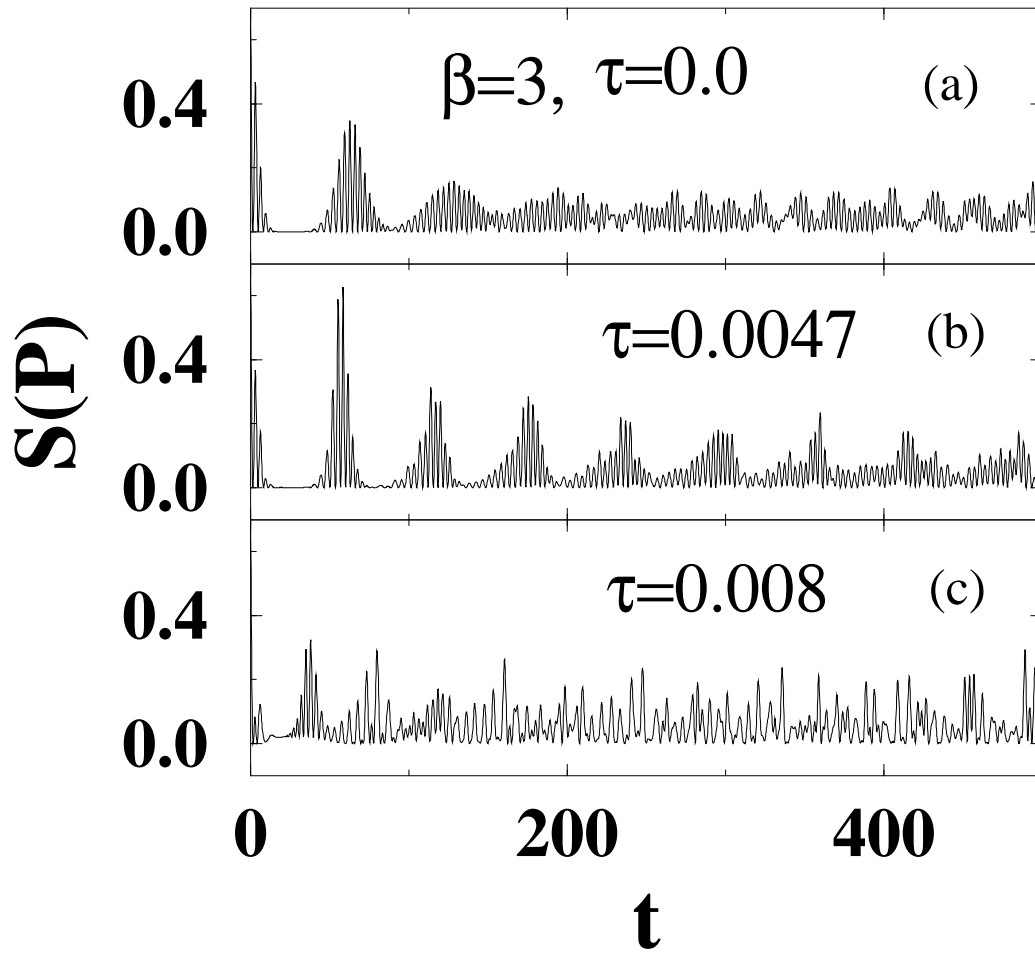


Fig. 2. Partial mutual quantum entropy $S(P)$ versus t for $\beta = 3$ and $\tau = 0.0, 0.0047, 0.008$.

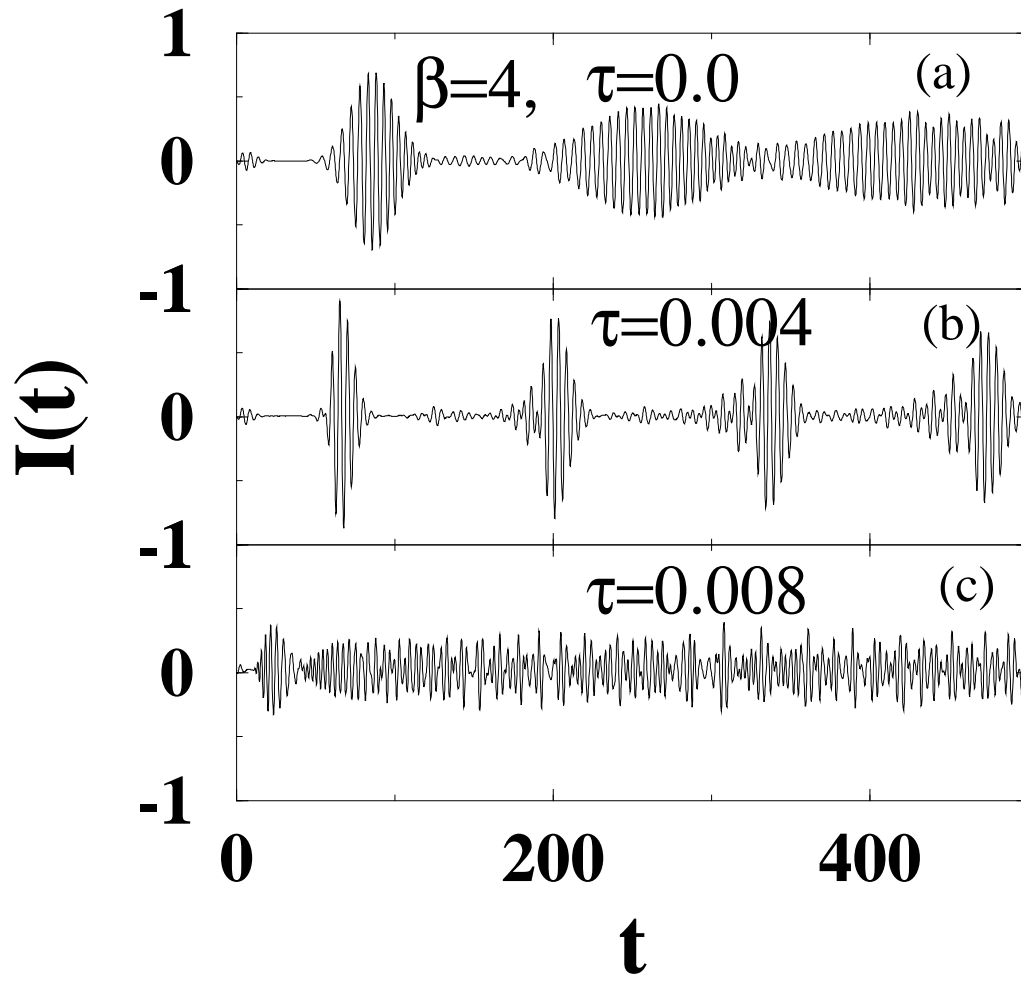


Fig. 3. Population Inversion $I(t)$ versus t for $\beta = 4$ and $\tau = 0.0, 0.004, 0.008$.

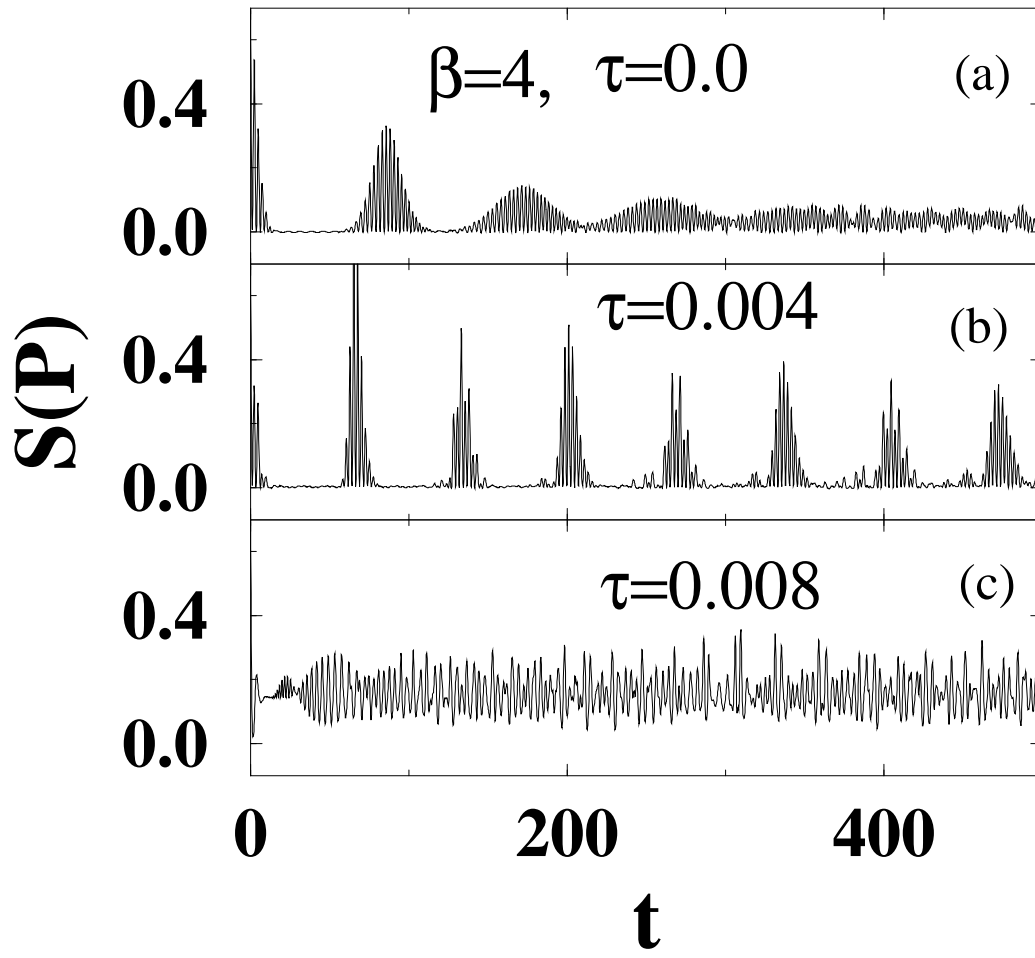


Fig. 4. Partial mutual quantum entropy $S(P)$ versus t for $\beta = 4$ and $\tau = 0.0, 0.004, 0.008$.

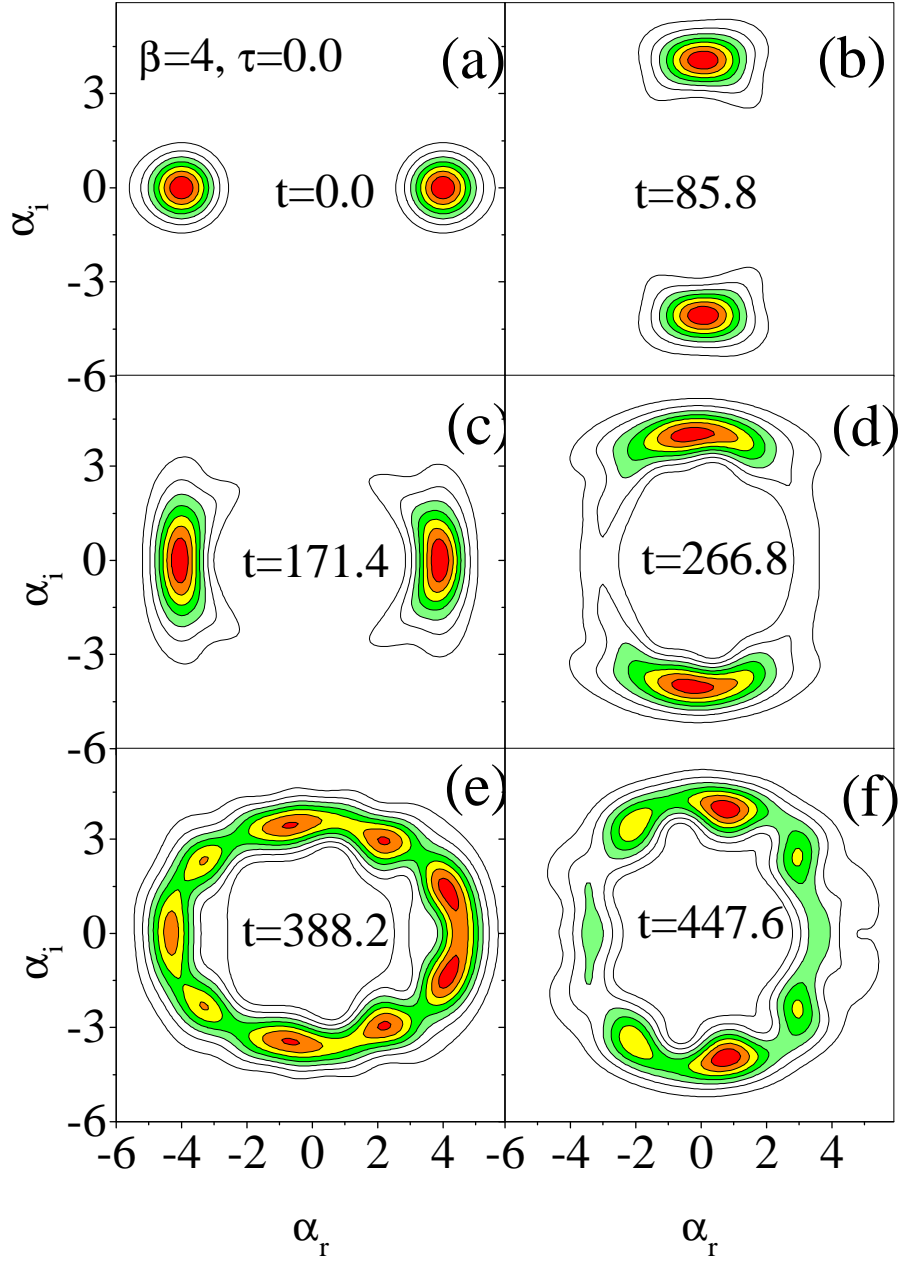


Fig. 5. Quasi-probability plots for $\beta = 4$ and $\tau = 0.0$ at $t = 0.0, 85.8, 171.4, 266.8, 388.2$ and 447.6 .

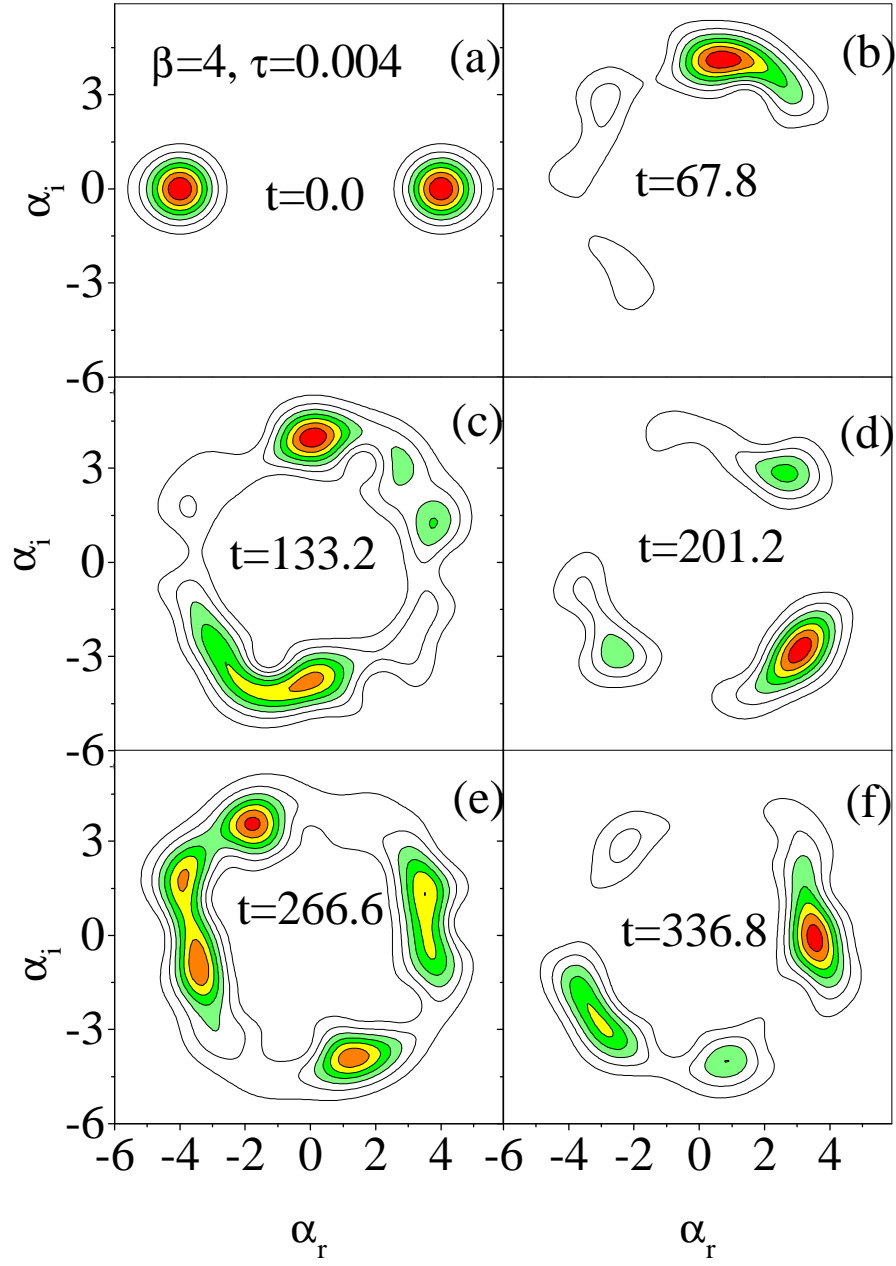


Fig. 6. Quasi- probability plots for $\beta = 4$ and $\tau = 0.004$ at $t = 0.0, 67.8, 133.2, 201.2, 266.6$, and 336.8 .